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EFFECT OF THE INTERRELATIONSHIP OF THERMAL AND MECHANICAL FIELDS ON THE PROPAGATION OF WAVE MOTIONS IN CUBICALLY ANISOTROPIC THERMOELASTIC MATERIALS

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A characteristic equation for a system of equations of motion of a cubically anisotropic medium with allowance for the relaxation time of thermal disturbances has been obtained, and expressions for the velocities of propagation of modified elastic and thermal waves have been found. The surfaces of inverse velocities have been constructed and the influence of the effect of interrelationship of thermal and mechanical fields on the change in the phase velocities of propagation of a quasilongitudinal elastic wave and a thermal wave in different planes of a cubically anisotropic material has been analyzed.

Introduction. The regularities of the propagation of plane waves and discontinuity surfaces in isotropic and anisotropic media whose thermal properties are described by a generalized (hyperbolic) heat-conduction law have been the focus of quite numerous works [1–3]. Below, we present results of implementing the method of characteristics of the theory of partial differential equations [4, 5] as applied to a system of equations of motion of a thermoelastic cubically anisotropic medium with account for the relaxation time of thermal disturbances.

Characteristic Equation. The resolving system of differential equations for thermoelastic anisotropic materials of a cubic symmetry system will be represented, according to [3], in the form

$$\left(A_4\Delta + (A_1 - A_2 - 2A_4)\,\partial_i^2\right)u_i + (A_2 + A_4)\,\partial_i \sum_{k=1}^{3} \partial_k u_k = \rho\partial_t^2 u_i + \beta\partial_i T\,,$$
$$c_\varepsilon \partial_t T + T_0\beta \sum_{k=1}^{3} \partial_t\partial_k u_k = -\sum_{k=1}^{3} \partial_k q_k\,,\quad \tau\partial_t q_i + q_i = -\lambda\partial_i T\,,\quad i = \overline{1, 3}\,. \tag{1}$$

We specify the initial conditions to system (1) on the surface $z(x_1, x_2, x_3, t) = 0$ and pass to new variables z, z_1, z_2, and z_3 according to the following scheme:

$$g = z\,(x_1, x_2, x_3, t)\,,\quad g_i = z_i\,(x_1, x_2, x_3, t)\,,\quad i = \overline{1, 3}\,.$$

Then

$$u_i\big|_{z=0} = f_i^{(1)}\,(z_1, z_2, z_3)\,,\quad \left.\frac{\partial u_i}{\partial z}\right|_{z=0} = f_i^{(2)}\,(z_1, z_2, z_3)\,,$$
$$T\big|_{z=0} = f^{(3)}\,(z_1, z_2, z_3)\,,\quad q_i\big|_{z=0} = f_i^{(4)}\,(z_1, z_2, z_3)\,,\quad i = \overline{1, 3}\,. \tag{2}$$

We substitute the relations for the derivatives of first and second orders in variables x_1, x_2, x_3, and t, which have been expressed by the variables z, z_1, z_2, and z_3, into the system of equations (1) and write the equations containing partial derivatives of second order g. As a result we obtain

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$$\left(A_4\sum_{k=1}^{3}\left(\frac{\partial z}{\partial x_k}\right)^2+(A_1-A_2-2A_4)\left(\frac{\partial z}{\partial x_i}\right)^2-\rho\left(\frac{\partial z}{\partial t}\right)^2\right)\frac{\partial^2 u_i}{\partial g^2}+(A_2+A_4)\frac{\partial z}{\partial x_i}\sum_{k=1}^{3}\frac{\partial z}{\partial x_k}\frac{\partial^2 u_k}{\partial g^2}-\beta\frac{\partial T}{\partial g}\frac{\partial z}{\partial x_i}+\ldots=0\,,$$

$$c_\varepsilon\frac{\partial T}{\partial g}\frac{\partial z}{\partial t}+T_0\beta\frac{\partial z}{\partial t}\sum_{k=1}^{3}\frac{\partial z}{\partial x_k}\frac{\partial^2 u_k}{\partial g^2}+\sum_{k=1}^{3}\frac{\partial z}{\partial x_k}\frac{\partial q_k}{\partial g}+\ldots=0\,,\quad \tau\frac{\partial z}{\partial t}\frac{\partial q_i}{\partial g}+\lambda\frac{\partial z}{\partial x_i}\frac{\partial T}{\partial g}+\ldots=0\,,\quad i=\overline{1,3}\,. \tag{3}$$

The partial derivatives $\frac{\partial^2 u_i}{\partial g^2}$ and $\frac{\partial T}{\partial g}$, $\frac{\partial q_i}{\partial g}$, $i=\overline{1,3}$ can have weak discontinuities on the surface $z(x_1, x_2, x_3, t) = 0$ only in the case where the equality of the determinant composed of the coefficients of these derivatives to zero holds. After simple manipulations, we represent the characteristic determinant in the following form:

$$\det\left\|\, w_{ij}\right\|_{7\times 7}=0\,, \tag{4}$$

where $w_{ii} = p_i^2 + (\delta_1 - p_i^2)b - \frac{p_0^2}{c_1^2}$, $w_{ij} = (a+b)p_ip_j$, $w_{i+3,i+3} = n_* p_0$, $w_{i7} = -p_i$, $w_{i+3,7} = w_{7,i+3} = p_i$, $w_{7i} = \varepsilon p_0 p_i$, and $w_{77} = p_0$; $i\neq j=\overline{1,3}$.

Expanding the determinant (4), we obtain

$$p_0^2\left(\frac{k_0p_0^8}{c_1^8}+\frac{k_1p_0^6}{c_1^6}+\frac{k_2p_0^4}{c_1^4}+\frac{k_3p_0^2}{c_1^2}+k_4\right)=0\,. \tag{5}$$

Here we have introduced the following notation:

$$k_0=-n_*\,,\quad k_1=(1+n_*(1+2b+\varepsilon))\,\delta_1\,, \tag{6}$$

$$\begin{gathered}k_2=-\left(1+b^2n_*+2b\,(1+n_*+\varepsilon n_*)\right)\left(\delta_1^2-2\delta_2\right)\\-\left(2+2b^2n_*+b\,(4-2\,(a-1)\,n_*)-(a-1)\,n_*\,(1+a+2\varepsilon)\right)\delta_2\,;\end{gathered} \tag{7}$$

$$\begin{gathered}k_3=b\left(2+b\,(1+n_*+\varepsilon n_*)\right)\left(\delta_1^3-3\delta_1\delta_2+3\delta_3\right)\\+\left(4b^3n_*+6\,(a-1)\,(an_*+\varepsilon n_*-1)+6b^2\,(an_*+\varepsilon n_*+1)\right.\\\left.-(a-1)\left(3+n_*+3\varepsilon n_*-2a^2n_*+a\,(3+n_*-3\varepsilon n_*)\right)\right)\delta_3\\+\left(1-a^2\,(1+bn_*)+b^2\,(3+n_*-\varepsilon n_*)-2ab\,(1+bn_*+\varepsilon n_*)+b\,(4+n_*+2\varepsilon n_*)\right)(\delta_1\delta_2-3\delta_3)\,,\end{gathered} \tag{8}$$

$$\begin{gathered}k_4=-b^2\left(\delta_1^4-4\delta_1^2\delta_2+2\delta_2^2+4\delta_1\delta_3\right)+(a-1)\,(1+a+2b)\,b\\\times\left(\delta_1^2\delta_2-2\delta_2^2-\delta_1\delta_3\right)+2b\left(a^2-1-b+2ab\right)\left(\delta_2^2-2\delta_1\delta_3\right)\\-\left(1+2a^3+2b+2ab\,(b-3)+2b^2\,(1+2b)+a^2\,(4b-3)\right)\delta_1\delta_3\,.\end{gathered} \tag{9}$$

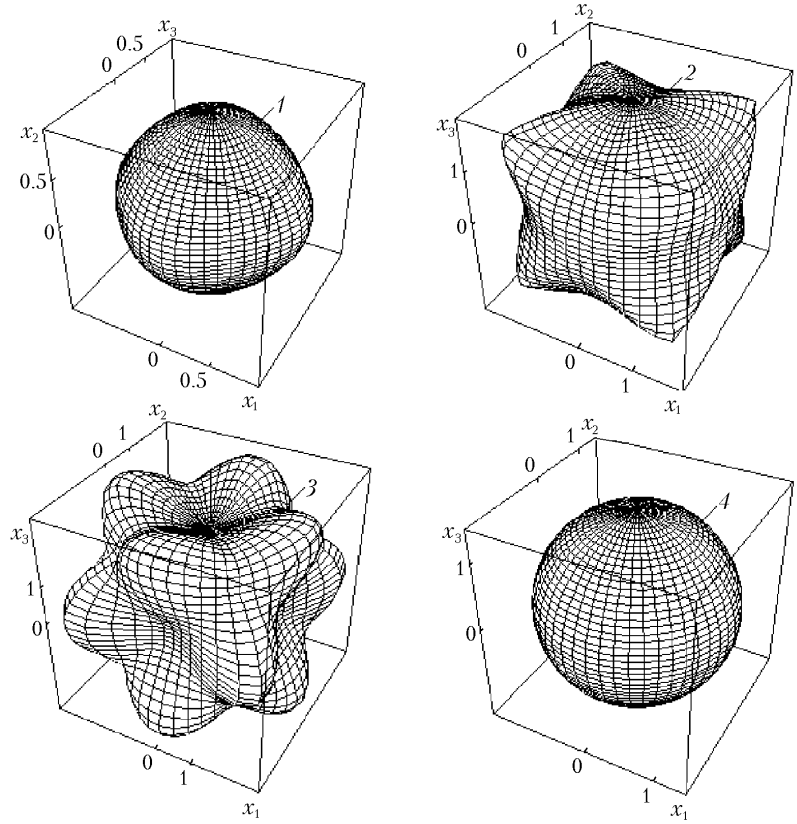

Fig. 1. Surfaces of inverse velocities for modified elastic and thermal waves propagating with velocities: 1) v_1, 2) v_2, 3) v_3, and 4) v_4.

The characteristic equation (5) yields the existence of a stationary discontinuity surface, three modified elastic waves whose propagation is affected by a temperature field, and the modified thermal wave whose propagation is accompanied by elastic deformations.

Velocities of Propagation of Waves. Taking into account that the velocity of propagation of the discontinuity surface is $V^2 = p_0^2/\delta_1$ [4], after simple manipulsations, we represent Eq. (5) in the form

$$k_0 v^8 + \hat{k}_1 v^6 + \hat{k}_2 v^4 + \hat{k}_3 v^2 + \hat{k}_4 = 0 \,. \tag{10}$$

Here the coefficients $\hat{k}_s$ are obtained from formulas (6)–(9) for the coefficients k_s, $s = \overline{1, 4}$, by replacement of the symmetric polynomials δ_1, δ_2, and δ_3 by $\hat{\delta}_1$, $\hat{\delta}_2$, and $\hat{\delta}_3$. Expressions for velocities will be found using the cubic resolvent [6] of Eq. (10). As a result we obtain

$$v_i = \sqrt{\frac{1}{2}\left(\delta^{(i)}\sqrt{z_0} - (-1)^i \sqrt{-2\delta^{(i)} q - \sqrt{z_0}\,(2p + z_0)}\right)} \,, \tag{11}$$

where

$$z_0 = \sqrt{2\sqrt{-\frac{P}{3}}\cos(\Lambda) - \frac{2p}{3}}\,;\quad \Lambda = \frac{1}{3}\arccos\left(-\frac{Q}{2}\sqrt{-\left(\frac{3}{P}\right)^3}\right);\quad P = -\frac{p^2 + 12r}{3}\,;\quad Q = \frac{8pr}{3} - \frac{2p^3}{27} - q^2\,;$$

$$p = \frac{1}{k_0}\left(\hat{k}_2 - \frac{3\hat{k}_1^2}{8}\right);\quad q = \frac{1}{k_0}\left(\frac{\hat{k}_1^3}{8} - \frac{\hat{k}_1\hat{k}_2}{2} = \hat{k}_3\right);\quad r = \frac{1}{k_0}\left(\frac{\hat{k}_1^2\hat{k}_2}{16} - \frac{\hat{k}_1\hat{k}_3}{4} - \frac{3\hat{k}_1^4}{256} + \hat{k}_4\right);\quad \delta^{(i)} = (\delta_{i1} + \delta_{i2} - \delta_{i3} - \delta_{i4})\,;$$

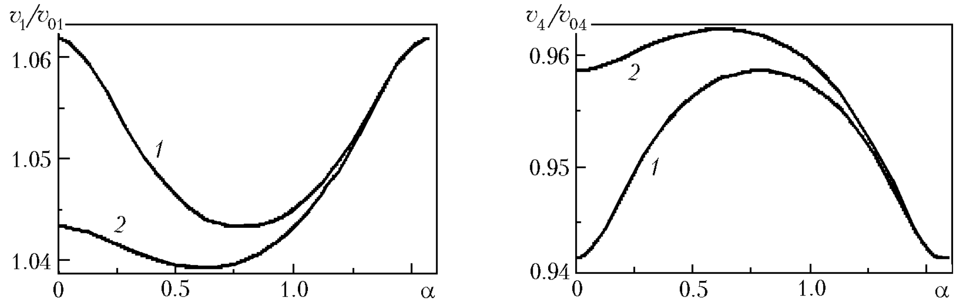

Fig. 2. Velocity ratios v_1/v_{01} and v_4/v_{04} vs. slope of the normal to the characteristic surface for different planes of a cubically anisotropic material: 1) plane $x_3 = 0$; 2) plane $\hat{x}_2 = 0$.

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ and $i = \overline{1, 4}$.

We note that a modified quasilongitudinal elastic wave propagates with a velocity v_1, modified quasitransverse elastic waves propagate with velocities v_2 and v_3, and a modified thermal wave propagates with a velocity v_4.

Velocity Surfaces and Their Sections. Formulas (11) enable us to construct inverse-velocity surfaces (or velocity surfaces) and to characterize the dependences of the velocities of propagation of thermoelastic waves v_1 (phase velocity) on the slope of the normal to the characteristic surface. Figure 1 gives the inverse-velocity surfaces for modified elastic and thermal waves propagating in lead ($a = 0.84$, $b = 0.31$, $\varepsilon = 0.609$, and $n = 1.7$; the numerical data have been taken from [7–9]). To find the absolute values of the velocities of propagation of the waves we must multiply the reciprocals of the relative values by $c_1 = \sqrt{A_1/\rho}$.

From Fig. 1, it is clear that the surface of inverse velocities of the modified thermal wave propagating with a velocity v_4 is virtually no different from a sphere, i.e., the phase velocity of this wave is independent of the slope of the normal to the characteristic surface. The surfaces of inverse velocities of the quasitransverse waves contain convexity and concavity portions, which points to the occurrence of lacunas on the wave front. For quantitative analysis of the influence of the interrelationship of a deformation field and a temperature field on the propagation of wave motions in cubically anisotropic materials we consider the sections of inverse-velocity surfaces by coordinates planes and by planes passing through the origin of coordinates. Figure 2 gives the ratios of the velocities of the modified elastic v_1/v_{01} and thermal waves v_4/v_{04} propagating in the coordinate plane $x_3 = 0$ and in the plane $\hat{x}_2 = 0$ passing through the bisector of the coordinate angle $x_1 0 x_2$ and the coordinate axis x_3 as functions the slope of the normal to the characteristic surface (v_{01} and v_{04} are the velocities of propagation of the quasilongitudinal elastic and thermal waves respectively). In constructing, we use the numerical data given above.

It is noteworthy that the dependences of the velocity ratios v_1/v_{01} and v_4/v_{04} in the coordinate planes $x_1 = 0$ and $x_2 = 0$ have the same form as those in the plane $x_3 = 0$. From Fig. 2, it is clear that the appearance of a thermal field causes the velocity of propagation of the quasilongitudinal elastic wave to increase and the velocity of propagation of the thermal wave to decrease. The increase in the velocity is the largest (≈6%) in the coordinate plane $x_3 = 0$; the minimum (≈3.5%) change in the velocities v_1 and v_4 is observed in the plane $\hat{x}_2 = 0$. No change in the velocities of propagation of the quasitransverse waves in the coordinate planes and in the plane $\hat{x}_2 = 0$ due to the action of the temperature field has been found.

CONCLUSIONS

The expressions obtained for velocities enable us to evaluate the influence of the relaxation time of thermal disturbances on the velocities of propagation of elastic and thermal waves with allowance for the interrelationship of mechanical and temperature fields. An analysis of the dependences of the velocities v_1 and v_4 on the slope of the normal to the characteristic surface, which has been made for different planes of a cubically anisotropic material at different times of relaxation of thermal disturbances ($n_* = 1 \ldots 2$), has shown that the parameter n_* most substantially affects the velocity of propagation of a modified quasilongitudinal wave.

NOTATION

A_1, A_2, and A_4, elasticity constants; $a = A_2/A_1$, $b = A_4/A_1$, and $c_1 = \sqrt{A_1/\rho}$; c_ε, specific heat at constant deformation; $n_i = \cos\alpha$, direction cosines of the slope of the normal to the characteristic surface; $n_* = \tau\omega_*$, characteristic number of vibrations; $p_i = \partial z/\partial x_i$, $p_0 = \partial z/\partial t$, and q_i, components of the heat-flux vector; T, change in the absolute temperature; T_0, initial temperature; t, time variable; u_i, displacement-vector components; $v = V/c_1$, dimensionless velocity of propagation of the discontinuity surface; Δ, Laplace operator; α, angle between the normal to the characteristic surface and the coordinate axis; α_{th}, coefficient of linear thermal expansion; $\beta = (A_1 + 2A_2)\alpha_{th}$; $\delta_1 = p_1^2 + p_2^2 + p_3^2$; $\delta_2 = p_1^2p_2^2 + p_3^2p_2^2 + p_1^2p_3^2$; $\delta_3 = p_1^2p_2^2p_3^2$; $\hat{\delta}_1 = n_1^2 + n_2^2 + n_3^2$; $\hat{\delta}_2 = n_1^2n_2^2 + n_3^2n_2^2 + n_1^2n_3^2$; $\hat{\delta}_3 = n_1^2n_2^2n_3^2$; $\varepsilon = T_0\beta^2/(A_1c_\varepsilon)$, dimensionless connectivity coefficient; λ, thermal conductivity; ρ, density; τ, relaxation time of thermal disturbances; $\omega_* = c_\varepsilon A_1/\lambda$, characteristic quantity having the dimensions of a particle; $\partial_i = \partial/\partial x_i$ and $\partial_t = \partial/\partial t$. Subscript: th, thermal.

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